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Numerical solution of Compton source term in Fokker-Planck equation: Application to radiotherapy

Taposh Kumar Das *

Abstract: *In this article we adopt the numerical solution and calculating time of Compton source term of the Fokker-Planck equation. The calculation time is much important for dose calculation of radiotherapy and it is mostly dependent on the calculating time of Compton source term. The Fokker-Planck equation is the approximation of the Boltzmann transport equation for electrons. The number of electrons are needed to calculate the dose of radiotherapy for cancer treatment. We get this number of electrons by solving the Fokker-Planck equation.*

Keywords: *Compton source term, Boltzmann Transport equation, Fokker-planck, Radiotherapy, Dose calculation, Differential cross section, Møller coefficient, Mott coefficient, stopping power*

Introduction

For cancer treatment the high energy photon radiotherapy is very much useful in the present time. So, it is most important to calculate the expected dose distribution, before start the treatment of the patient, i.e., the distribution of absorbed radiative energy in the patient, has to be calculated. It depends on a CT scan (CT= computed tomography) of the tumour region and the treatment plan is tailored to the local geometry of the tumour and the surrounding tissue to guarantee the best possible dose. If the dose of radiotherapy in the tumour tissue is not very low then we can expect a curative effect. But if the dose is so high then the many healthy tissue surrounding the tumour will be destroyed or they will not be able to protect or avoid the undesirable side effect from the high dose. Therefore, one of the main parts for a treatment plan is the perfect dose calculation before beginning the treatment for effective the real treatment.

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We can find the exact dose calculation for photon and electron radiation by well known physical principles of interaction of radiation with human tissue by the transport of energy into the patients body that can be modelled and calculated by an appropriate Monte Carlo (MC) algorithm (Andreo P,1991). If we work carefully then it leads to exact results of the dose distribution in arbitrary geometries and nowadays highly developed MC codes for dose calculations are available but the computational time is very high in this case. Therefore this process is going to unattractive position day by day in clinical use.

The other alternative approach to circumvent the drawback of the MC codes called kernel models (Ahnesjö and Aspradakis,1999) offer a reliable and fast alternative for most types of radiation treatment. The pencil beam models are probably most in use and these models are based on the Fermi-Eyges theory of radiative transfer (Rossi and Greisen,1941) & (Eyges, 1948). Originally introduced for pure electron radiation by Hogstramet Mills and Almond (1981) and later generalized to photon radiation by (Gustafsson, Lind and Brahme, 1994)& (Ulmer and Harder, 1995), too. Although the result was good but the pencil beam models fail in complicated setting like air cavities or other inhomogeneities.

The third access for dose calculation attracted in the last few years is the deterministic Boltzmann equation of radiative transfer based on the physical interactions of radiation in tissue. A mathematical model can be developed that allows in principle an exact dose calculation like as MC models. The recent studies for pure electron radiation were mostly done by Börgers and E.W.Larsen (1996). Electron and combined photon and electron radiation were studied by Tervo et. al.(1999). Tervo and Kolmonen(2002) in the context of inverse therapy planning and Zhengming et al(2004) restricted their model to one dimensional slab geometry. The Boltzmann Transport equation (BTE) for photons are already solved numerically by Taposh (2012). It is very difficult to solve the Boltzmann Transport equation for electrons. Therefore we choose the Fokker planck approximation of the BTE for electrons. The compton source term is the part of the Fokker planck equation.

In this paper we represent the numerical result and the calculating time of the Compton source term. Here the time is very important for the dose calculation of radiotherapy because the time of dose calculation is mostly dependent on the calculating time of the Compton source term.

The Boltzmann model for coupled photon and electron transport

(Hartmut et.al.2006)

The photons and electrons move with high velocities so all the process can be regarded as time independent and the all calculations are done relativistic using the relativistic formulae for energy and fully relativistic scattering cross section. For convenience all energies are scaled by the rest energy of the electron $mc^2 = 0.511$ MeV, m being the

rest mass of the electron c is the velocity of light.

Let $\psi_\gamma(\mathbf{r}, \Omega_\gamma, \epsilon_\gamma) \cos \Theta dA d\Omega_\gamma d\epsilon_\gamma dt$ be the number of photons that move in time dt through area dA into the element of solid angle $d\Omega_\gamma$ around Ω_γ with an energy in the interval $(\epsilon_\gamma, \epsilon_\gamma + d\epsilon_\gamma)$. Θ is the angle between direction Ω_γ and outer normal of dA . $\Omega_\gamma = (\sin \varphi_\gamma \cos \vartheta_\gamma, \sin \varphi_\gamma \sin \vartheta_\gamma, \cos \varphi_\gamma)^T$ where φ_γ is the zenith angle and ϑ_γ is the polar angle in a Cartesian coordinate system. A similar definition holds for $\varphi_c(\mathbf{r}, \Omega_c, \epsilon_c) \cos \Theta dA d\Omega_c d\epsilon_c dt$ being the respective phase space fluence of electron, ϵ_c their kinetic energy and Ω_c their direction of flight. The transport equation for photons is

$$\begin{aligned} \Omega_\gamma \cdot \nabla \psi_\gamma(\mathbf{r}, \Omega_\gamma, \epsilon_\gamma) = & \rho_e(r) \int_0^\infty \int_{S^2} \tilde{\sigma}_{C,\gamma}(\epsilon'_\gamma, \epsilon_\gamma, \Omega'_\gamma \cdot \Omega_\gamma) \psi_\gamma(\mathbf{r}, \Omega'_\gamma, \epsilon'_\gamma) d\Omega'_\gamma d\epsilon'_\gamma \\ & - \rho_e(r) \sigma_{C,\gamma}^{\text{tot}}(\epsilon_\gamma) \psi_\gamma(\mathbf{r}, \Omega_\gamma, \epsilon_\gamma) \end{aligned} \quad (1)$$

where ρ_e is the electrons density of the medium and $\tilde{\sigma}_{c,\gamma}$ is the scattering cross section of the photons, differential in angle and energy for comptons scattering of photons and $\sigma_{C,\gamma}^{\text{tot}}(\epsilon_\gamma)$ is the total compton scattering cross section of photons.

The transport equation for electrons is

$$\begin{aligned} \Omega_c \cdot \nabla \psi_c(\mathbf{r}, \Omega_c, \epsilon_c) = & \rho_c(r) \int_0^\infty \int_{S^2_{\frac{1}{2}}} \tilde{\sigma}_{C,c}(\epsilon'_c, \epsilon_c, \Omega'_c \cdot \Omega_c) \psi_c(\mathbf{r}, \Omega'_c, \epsilon'_c) d\Omega'_c d\epsilon'_c \\ & + \rho_c(r) \int_{\epsilon_s}^\infty \int_{S^2_{\frac{1}{4}}} \tilde{\sigma}_M(\epsilon'_e, \epsilon_c, \Omega'_e \cdot \Omega_c) \psi_c(\mathbf{r}, \Omega'_e, \epsilon'_e) d\Omega'_e d\epsilon'_e \\ & + \rho_c(r) \int_{\epsilon_s}^\infty \int_{S^2_{\frac{2}{4}}} \tilde{\sigma}_{M,\delta}(\epsilon'_e, \epsilon_c, \Omega'_e \cdot \Omega_c) \psi_c(\mathbf{r}, \Omega'_e, \epsilon'_e) d\Omega'_e d\epsilon'_e \\ & + \rho_c(r) \int_{S^2} \sigma_{\text{Mott}}(\mathbf{r}, \epsilon_e, \Omega'_e \cdot \Omega_c) \psi_e(\mathbf{r}, \Omega'_e, \epsilon_e) d\Omega'_e \\ & - \rho_c(r) \sigma_M^{\text{tot}}(\epsilon_c) \psi_c(\mathbf{r}, \Omega_c, \epsilon_c) \\ & - \rho_c(r) \sigma_{\text{Mott}}^{\text{tot}}(\mathbf{r}, \epsilon_c) \psi_c(\mathbf{r}, \Omega_c, \epsilon_c) \end{aligned} \quad (2)$$

where ρ_c is the density of atomic cores in the medium and $\tilde{\sigma}_{C,e}$ is the scattering cross section of electrons differential in angle and energy for Compton scattering of electrons. $\tilde{\sigma}_M$ is the scattering cross section for primary electrons differential in angle and energy for Møller scattering and $\tilde{\sigma}_{M,\delta}$ is the cross section for secondary electrons ('delta-rays'). The scattering cross section for Mott scattering σ_{Mott} is only differential in angle, because Mott scattering is an elastic scattering. The total cross section for Møller and Mott scattering are σ_M^{tot} and $\sigma_{\text{Mott}}^{\text{tot}}$ respectively.

The angular integration in the electron equation are restricted by the kinematics of the scattering events. We define

$$\int_{S_{\frac{1}{2}}^2} f(\varphi, \vartheta) d\Omega := \int_0^{2\pi} \int_0^{\frac{\pi}{2}} f(\varphi, \vartheta) \sin \vartheta d\vartheta d\varphi$$

$$\int_{S_{\frac{1}{4}}^2} f(\varphi, \vartheta) d\Omega := \int_0^{2\pi} \int_0^{\frac{\pi}{4}} f(\varphi, \vartheta) \sin \vartheta d\vartheta d\varphi$$

$$\int_{S_{\frac{3}{4}}^2} f(\varphi, \vartheta) d\Omega := \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(\varphi, \vartheta) \sin \vartheta d\vartheta d\varphi$$

where the axes of reference are defined by Ω_γ and Ω_e in (1) and (2) respectively.

The energy range will be from 0 to infinite. We denote the patients surface Ω and subdivided it into a part that is irradiate by photons and electrons Γ and the rest of the surface Λ (i.e., $\Omega = \Gamma \cup \Lambda$).

we assume that the energy spectrum and the angle distribution of photons and electrons are known on Γ . Then the boundary condition are

$$\left. \begin{array}{ll} \psi_\gamma(\mathbf{r}, \boldsymbol{\Omega}_\gamma, \epsilon_\gamma) |_\Gamma = \Psi_\gamma^0(S, \boldsymbol{\Omega}_\gamma, \epsilon_\gamma); & \text{for } n_\Gamma \cdot \boldsymbol{\Omega}_\gamma < 0 \\ \psi_\gamma(\mathbf{r}, \boldsymbol{\Omega}_\gamma, \epsilon_\gamma) |_\Lambda = 0; & \text{for } n_\Lambda \cdot \boldsymbol{\Omega}_\gamma < 0 \\ \psi_e(\mathbf{r}, \boldsymbol{\Omega}_e, \epsilon_e) |_\Gamma = \Psi_e^0(S, \boldsymbol{\Omega}_e, \epsilon_e); & \text{for } n_\Gamma \cdot \boldsymbol{\Omega}_e < 0 \\ \psi_e(\mathbf{r}, \boldsymbol{\Omega}_e, \epsilon_e) |_\Lambda = 0; & \text{for } n_\Lambda \cdot \boldsymbol{\Omega}_e < 0 \end{array} \right\} \quad (3)$$

n_Γ and n_Λ begin the outer normal of Γ and Λ respectively. S is the position in the two dimensional surface that accounts for intensity modulation in the irradiated surface. Boundary conditions are only formulated for fluencies that are going into the patients body, because multiple scattering inside the body can lead to outward fluencies.

The transport equation (1),(2) and the boundary condition (3) model the transport of photons and electrons in heterogeneous biological tissue. Different media like muscle and bone are modelled by different densities of electrons ρ_e , atomic cross ρ_c and mott cross sections(which depend on the atomic number $Z=Z(\mathbf{r})$ of the irradiate media). So heterogeneous media are included into the model from the beginning and will be kept in mind in later asymptotic development.

Absorbed dose

From the above discussion we have seen only one type of scattering events leading to energy deposition in the irradiated medium namely Møller scattering. We see that by a Møller scattering event a free electron transfers energy and momentum to a bound electron, so that the latter can leave the molecule and then the binding energy of the electron has been transferred to the molecule. We have seen the energy balance in below: let ϵ'_e be the kinetic energy of primary electron before scattering and an electron at rest his kinetic energy 0. Let, after scattering, the kinetic energy of a primary electron,

secondary electron and an ionized molecule are ϵ_e , ϵ_δ and ϵ_B respectively. Therefore the energy balance will be;

$$\epsilon'_e = \epsilon_e + \epsilon_\delta + \epsilon_B \tag{4}$$

We can not understand which electron is primary or which electron is secondary. But by definition after scattering, the energy of primary electron is greater than the energy of the secondary electrons, so we can write the following two equations:

$$\epsilon_e \geq \frac{\epsilon'_e - \epsilon_B}{2} \tag{5}$$

$$\epsilon_\delta \leq \frac{\epsilon'_e - \epsilon_B}{2} \tag{6}$$

we consider the lower bound of energy of electron ϵ_s that is also relevant for dose calculation. Also we have to study the energy transfer process in a Møller scattering event and to sum over all possible combinations of outgoing and incoming energies and direction of electrons in Møller scattering for calculating the absorbed dose. Here we have two different contributions to the absorbed dose:

(i) If $\epsilon_\delta \leq \epsilon_s$ or $\epsilon_\delta \in [0, \epsilon_s]$, i.e., delta electrons will not transport and it absorbed locally. So, we see that the ionization energy of the molecules ϵ_B and the kinetic energy ϵ_δ of the secondary electrons contributes to the absorbed dose. Therefore, the energy loss ($\epsilon'_e - \epsilon_e$) of the primary electrons enters into the dose formula. The limit of the incoming electrons, $\epsilon'_e \in [\epsilon_s, \infty)$, and for the scattered primary electrons, $\epsilon_e \in [\epsilon'_e - \epsilon_s - \epsilon_B, \epsilon'_e - \epsilon_B]$ because we have got from eq(4) $\epsilon_e = \epsilon'_e - \epsilon_\delta - \epsilon_B$ where $\epsilon_\delta \in [0, \epsilon_s]$.

(ii) If $\epsilon_\delta \geq \epsilon_s$ or $\epsilon_\delta \in [\epsilon_s, \frac{\epsilon'_e - \epsilon_B}{2}]$, i.e., delta electrons are transported, so in this case only the ionization energy ϵ_B of the molecules contributes in the absorbed dose, i.e., $\epsilon_B = [\epsilon'_e - \epsilon_e] - \epsilon_\delta$ enters in to the dose formula.

The limit of the energy of the incoming electrons is $\epsilon'_e \in [\epsilon_s, \infty)$. We get from eq(4) $\epsilon_e = \epsilon'_e - \epsilon_\delta - \epsilon_B$. Here, $\epsilon_\delta \in [\epsilon_s, \frac{\epsilon'_e - \epsilon_B}{2}]$. Therefore we get the energy limit for the scattered primary electrons, $\epsilon_e \in [\frac{\epsilon'_e - \epsilon_B}{2}, \epsilon'_e - \epsilon_s - \epsilon_B]$.

Therefore, we can get the following exact formula for the absorbed dose:

$$D(r) = \frac{mc^2 \rho_e(r)}{\rho(r)} T \int_{s^2} \int_{s^2_{\frac{1}{4}}} \int_{\epsilon_s}^{\infty} \int_{\epsilon'_e - \epsilon_s - \epsilon_B}^{\epsilon'_e - \epsilon_B} [\epsilon'_e - \epsilon_e] \tilde{\sigma}_M(\epsilon'_e, \epsilon_e, \Omega'_e \cdot \Omega_e) \psi_e(r, \Omega'_e, \epsilon'_e) d\epsilon_e d\epsilon'_e d\Omega'_e d\Omega_e + \frac{mc^2 \rho_c(r)}{\rho(r)} T \int_{s^2} \int_{s^2_{\frac{1}{4}}} \int_{\epsilon_s}^{\infty} \int_{\frac{\epsilon'_e - \epsilon_B}{2}}^{\epsilon'_e - \epsilon_s - \epsilon_B} [\epsilon'_e - \epsilon_e - \epsilon_\delta] \tilde{\sigma}_M(\epsilon'_e, \epsilon_e, \Omega'_e \cdot \Omega_e) \times \psi_e(r, \Omega'_e, \epsilon'_e) d\epsilon_e d\epsilon'_e d\Omega'_e d\Omega_e \tag{7}$$

where mc^2 being the rest energy of the electron to rescale energy, ρ the local density of the medium and T is the duration of the irradiation. If all quantities are calculated in SI units this formula leads to SI unit Jkg^{-1} or Gray(Gy) for the dose.

By the asymptotic development we can write this equation in the following form;

$$D(r) = \frac{mc^2 \rho_c(r)}{\rho(r)} T \int_{s^2} \int_{s^2_{\frac{1}{4}}} \int_{\epsilon_s}^{\infty} \int_{\frac{\epsilon'_c - \epsilon_B}{2}}^{\epsilon'_c - \epsilon_c} [\epsilon'_c - \epsilon_c] \tilde{\sigma}_M(\epsilon'_c, \epsilon_c, \Omega'_c \cdot \Omega_c) \psi_c(r, \Omega'_c, \epsilon'_c) d\epsilon_c d\epsilon'_c d\Omega'_c d\Omega_c \quad (8)$$

To solve the equation (8) we need the number of electrons (ψ_e). By solving the equation (2) we can find these electrons. It is very difficult to solve this Boltzmann transport equation for electrons (2) directly. So we choose a Fokker-Planck development in the standard way (Pomraning, 1992) to find this number of electrons. The resulting asymptotic equation is

$$\begin{aligned} \Omega_e \cdot \nabla \psi_e(\mathbf{r}, \Omega_e, \epsilon_e) - [T_{Mott}(\mathbf{r}, \epsilon_e) + T_M(\mathbf{r}, \epsilon_e)] L \psi_e \\ - \frac{\partial}{\partial \epsilon_e} [S_M(\mathbf{r}, \epsilon_e) \psi_e] = Q(\mathbf{r}, \Omega_e, \epsilon_e), \end{aligned} \quad (9)$$

where L is the Laplace operator on the sphere which is given by

$$L = \frac{\partial}{\partial \mu_e} (1 - \mu_e^2) \frac{\partial}{\partial \mu_e} + \frac{1}{1 - \mu_e^2} \frac{\partial^2}{\partial \varphi_e^2}, \quad \mu_e = \cos \vartheta_e. \quad (10)$$

$$\text{where; } 0 \leq \vartheta_e \leq \pi. \quad (11)$$

Now we use the Fokker-Planck approximation that leads to the final result for the asymptotically developed absorbed dose:

$$D(r) = \frac{mc^2 \rho_c(r)}{\rho(r)} T \int_{\epsilon_s}^{\infty} S_M(\mathbf{r}, \epsilon'_c) \Phi(\mathbf{r}, \epsilon'_c) d\epsilon'_c \quad (12)$$

with $\Phi(\mathbf{r}, \epsilon_c) := \int_{s^2} \psi_e(\mathbf{r}, \Omega'_e, \epsilon_c) d\Omega'_e$.

Here, Q is the Compton source term (T.K.Das and S.U.Suma, 2013)

$$Q(\mathbf{r}, \Omega_e, \epsilon_e) = \rho_e(\mathbf{r}) \int_{D \subset S^2_{1/2}} \frac{\sigma_{C,e}(E'_\gamma, C) \psi_\gamma(\mathbf{r}, \Omega'_\gamma, E'_\gamma) [(1 - C^2) E'^2_\gamma + 2E'_\gamma + 1]^2}{4E'_\gamma C^2 (1 + E'_\gamma)} d\Omega'_\gamma, \quad (13)$$

where

$$C = C(\Omega'_\gamma) = \Omega_e \cdot \Omega'_\gamma \text{ and} \quad (14)$$

$$E'_\gamma = \frac{\epsilon_e + \sqrt{C^2 \epsilon_e (2 + \epsilon_e)}}{2C^2 - \epsilon_e + C^2 \epsilon_e}. \quad (15)$$

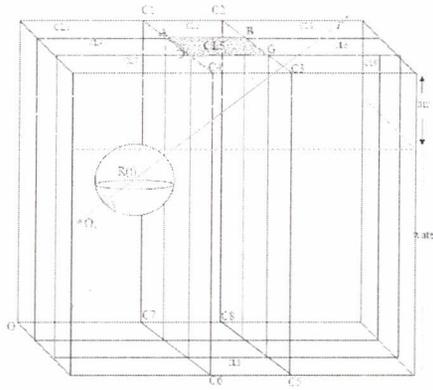


Figure 1

The Matlab code

We have made a MATLAB code to solve the Compton source term. This total code contains 16 m-files. We use the the following names for those m-files are PSI0, PSI1, PSI2, PSI3, PS1, PS2, PS3, H1, H2, H3, g0, g1, g2, CBP, EX and main2. We discuss about the m-files in the following:

1. In main2 we give the necessary data (input data) and get output in here.
 For input data we use some symbol in this m-file which are L, ED, R, OME, E, HW, Nx, Ny, Nx1, Ny1, Nx2, Ny2, NW1, NW2 and NW3. All the notations and the meaning of those are following:
 - (a) $L=[L_x, L_y, L_z]$, where L_x , L_y and L_z are the length of three edges of the cube (Figure 1).
 - (b) ED is the length of edge of irradiate square on the upper surface of the cube.
 - (c) R is the point inside the domain where we want to calculate the number of photons (in the Figure 1 point \mathbf{R} is $\mathbf{R}(t)$).
 - (d) We calculate the number of photons at a fixed direction. OME is the unit vector in that direction. In the Figure 1 it is Ω_γ .
 - (e) E is the energy of photon.
 - (f) In the cube (domain) we use two parts one of that is water and the other is air. HW is the length of height of the water part.

For calculating $\psi_\gamma^{(1)}$ (T.K. Das, 2012, p-60) we need to calculate $g_{\epsilon_\gamma, \Omega_\gamma}^{(0)}(\lambda)$ ((T.K. Das, 2012, p-66)) and in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(0)}(\lambda)$ we see that there are two integrations one for polar angle and one for zenith angle. Here we use y for polar angle and x for zenith angle.

- (g) Nx is the division number of the range of x of the domain of the integral in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(0)}(\lambda)$.
- (h) Ny is the division number of the range of y of the domain of the integral in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(0)}(\lambda)$.

For calculating $\psi_\gamma^{(2)}$ we need to calculate $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$ and in the formula of $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$ has two integrations one for polar angle and one for zenith angle. Here we use y1 for polar angle and x1 for zenith angle.

- (i) Nx1 is the division number of the range of x1 of the domain of the integral in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$.
- (j) Ny1 is the division number of the range of y1 of the domain of the integral in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$.
We use x2 and y2 for the zenith and polar angle respectively in the integral involving in the formula of $g_{\epsilon_\gamma, \Omega_\gamma}^{(2)}(\lambda)$ to calculate $\psi_\gamma^{(3)}$.
- (k) Nx2 is the division number of the range of x2 of the domain of the integral in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(2)}(\lambda)$.
- (l) Ny2 is the division number of the range of y2 of the domain of the integral in the formulation of $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$.

We observe that the value of $\psi_\gamma^{(1)}$ at a point inside air is much smaller than the value of $\psi_\gamma^{(1)}$ at a point inside water. It is almost zero. So, in our MATLAB code we use only the points which are inside in water to calculate $\psi_\gamma^{(i)}$, where $i = 1, 2, 3, \dots, M$ (T.K.Das, 2012, p-61).

We see in the Ph.D thesis paper (T.K.Das, 2012, p-61) there is an integration with respect to l and the limit of this integration is from 0 to λ . But by the above discussion we will take the part of λ which inside in the water.

- (m) NW1 is the number of divisions of the water part of λ which is the range of the integral domain involved in the formula of $\psi_\gamma^{(1)}$.
- (n) NW2 is the number of divisions of the water part of λ which is the range of the integral domain involved in the formula of $\psi_\gamma^{(2)}$.
- (o) NW3 is the number of divisions of the water part of λ which is the range of the integral domain involved in the formula of $\psi_\gamma^{(3)}$.

2. In CBP we find the boundary point \mathbf{r}^* (Figure 1).

3. In EX we obtain the exponential part of the formula of $\psi_\gamma^{(0)}(\mathbf{r}(\lambda), \Omega_\gamma, \epsilon_\gamma)$ (T.K.Das, 2012, p-60, Eq(4.12)). In the exponential part, there exist a line integral and the limit of this integral is from 0 to λ (λ is the distance from boundary point \mathbf{r}^* to $\mathbf{R}(t)$, see Figure 1). If the domain that contains the segment $[0, \lambda]$ is heterogeneous then the electron density is different for different media. In making this program we have taken this fact into consideration.

4. In PSI0, we solve the number of non-interacting photons by using CBP and EX.
5. In g0 we find $g_{\epsilon_\gamma, \Omega_\gamma}^{(0)}(\lambda)$ by using this PSI0. Here we use the support (mentioned in previous section) to calculate the integration of g0.
6. To calculate g0 we need the value of integrand at the limiting points of the domain of integral. In H1 we find this value.
7. In PSI1 we find the number of one time interacting photons by using g0 and EX.
8. To find PSI1, we need the value of the integrand involving in PSI1 at the limiting points of the domain of the integral. We calculate this value in PS1.
9. In g1 we find $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$ by using this PSI1. Here we use the support (mentioned in previous section) to calculate the integration of g1.
10. To calculate g1 we need the value of integrand involving in the formula of $g_{\epsilon_\gamma, \Omega_\gamma}^{(1)}(\lambda)$ at the limiting point of the integral domain. In H2 we find this value.
11. In PSI2 we find the number of two time interacting photons by using g1 and EX.
12. To find PSI2, we need the value of the integrand of the integration involving in PSI2 at the limiting points of the integrating domain. We have calculated that value in PS2.

Similarly, we calculate the g2, PSI3, PS3 and H3 like as g1, PSI2, PS2 and H2 respectively. In main2 we give the necessary data (input data) and get output from here.

Results and Discussions

To calculate the integration numerically we use the 2D composite trapezoidal rule. Here we check that our numerical code regarding this part works well.

Let N_x and N_y be the number of divisions of the range of x and y respectively. We give the following example.

We calculate by hand and see that the value of

$$\int_1^3 \int_1^3 (x^2 \cos(y) + y^2 \sin(x)) dx dy = 7.1928. \tag{16}$$

The Table 1 gives the numerical result of the integration (16) which is solved by 2D composite trapezoidal rule.

We have taken the unit vector

$$\Omega_\gamma = \left[0, \frac{-0.05}{1.5608}, \frac{-01.56}{1.5608} \right], \tag{17}$$

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Nx	Ny	Numerical value	Error
10	10	7.1977	0.0129
20	20	7.1896	0.0032
30	30	7.1914	.0014
40	40	7.1920	7.6061e-004
50	50	7.1923	4.7000e-004
60	60	7.1925	3.1217e-004
70	70	7.1926	2.1710e-004
80	80	7.1926	1.5525e-004
100	100	7.1927	8.2618e-005
150	150	7.1928	1.0890e-005

Table 1

Nx2	Ny2	Q Compton source term	Time
2	2	.3253	30 sec.
3	3	.7577	50 sec.
4	4	1.1291	75 sec.
5	5	1.229	100 sec.
6	6	1.2182	120 sec.
7	7	1.2093	209 sec.

Table 2

then the approximate value of the Compton source term at [1.5, 1.45, 1.44] going to, this direction are given in the following Table 2. We observe that the value of integral in the formula of the compton source term is a good approximation of the exact value for $(Nx, Ny, Nx1, Ny1, Nw1, Nw2) \pm (4, 7, 8, 8, 12, 1)$. All the mentioned cross sections and the stopping power have been shown in the appendices.

Conclusion

From the Table 2 we see that the approximate value of the compton source term is 1.2 and the calculating time is near 100 seconds (Row 5 of Table 2), which is not so high. From the equation (12) of dose calculation we see that the number of electrons is needed for the dose which we can get by solving the equation (2) of Boltzmann Transport equation for electrons. It is difficult to find the number of electrons by calculating the equation (2) therefore we have to choose a Fokker-Planck development in the standard way (Pomraning, 1992) to find this number of electrons.

This paper represent the approximate numerical value and calculating time of the

Compton source term of the Fokker-Planck development of the Boltzmann transport equation for electrons which is needed for the numerical solution of this Fokker-Planck development to calculate the dose of radiotherapy for cancer treatment.

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Appendices

We have used the Compton scattering cross section in our Boltzmann model for solving the system of photon equation. The differential scattering cross section are differential in energy and in solid angle. The Compton scattering cross section can be decomposed into a product of a cross section, that is only differential in solid angle or energy and a Dirac delta function. Total cross section is calculated by integrating the double differential cross section with respect to energy and solid angle. Because the Delta functions one integral is always trivial.

To represent the cross-section we have used the quantities with a prime for incoming particles and the quantities without prime for outgoing particles. We have used the following symbols;

- (a) for incoming energy we have used ϵ'_γ ;
- (b) for outgoing energy we have used ϵ_γ ;
- (c) for incoming direction of photon we have used Ω'_γ ;
- (d) for outgoing direction of photon we have used Ω_γ ;
- (e) for incoming direction of electron we have used Ω'_e ;
- (f) for outgoing direction of electron we have used Ω_e ;

Appendix A

Differential cross section for Compton scattering of photons

Literature: (C.M.Davisson, R.D.Evans, 1952)

$$\tilde{\sigma}_{C,\gamma}(\epsilon'_\gamma, \epsilon_\gamma, \mathbf{\Omega}'_\gamma \cdot \mathbf{\Omega}_\gamma) = \sigma_{C,\gamma}(\epsilon'_\gamma, \mathbf{\Omega}'_\gamma \cdot \mathbf{\Omega}_\gamma) \delta_{C,\gamma}(\epsilon'_\gamma, \epsilon_\gamma) \quad (0.1)$$

with

$$\sigma_{C,\gamma}(\epsilon'_\gamma, \mathbf{\Omega}'_\gamma \cdot \mathbf{\Omega}_\gamma) = \frac{r_e^2}{2} \left[\frac{1}{1 + \epsilon'_\gamma(1 - \cos \vartheta_\gamma)} \right]^3 \left[1 + \cos^2 \vartheta_\gamma + \frac{\epsilon_\gamma'^2(1 - \cos \vartheta_\gamma)^2}{1 + \epsilon'_\gamma(1 - \cos \vartheta_\gamma)} \right] \quad (0.2)$$

$$\delta_{C,\gamma}(\epsilon'_\gamma, \epsilon_\gamma) := \delta \left(\epsilon_\gamma - \frac{\epsilon'_\gamma}{1 + \epsilon'_\gamma(1 - \cos \vartheta_\gamma)} \right) \quad (0.3)$$

Total cross section for Compton scattering of photons

Literature: (Davisson and Evans, 1952)

$$\sigma_{C,\gamma}^{\text{tot}}(\epsilon_\gamma) = 2\pi r_e^2 \left[\frac{1 + \epsilon_\gamma}{\epsilon_\gamma^2} \left(\frac{2(1 + \epsilon_\gamma)}{1 + 2\epsilon_\gamma} - \frac{1}{\epsilon_\gamma} \ln(1 + 2\epsilon_\gamma) \right) + \frac{1}{2\epsilon_\gamma} \ln(1 + 2\epsilon_\gamma) - \frac{1 + 3\epsilon_\gamma}{(1 + 2\epsilon_\gamma)^2} \right] \quad (0.4)$$

Differential cross section for Compton scattering of electrons

Calculation based on (Davisson and Evans, 1952)

$$\tilde{\sigma}_{C,e}(\epsilon'_\gamma, \epsilon_e, \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e) = \sigma_{C,e}(\epsilon'_\gamma, \epsilon_e, \mathbf{\Omega}'_\gamma \cdot \mathbf{\Omega}_e) \delta_{C,e}(\epsilon'_\gamma, \epsilon_e) \quad (0.5)$$

with

$$\sigma_{C,e}(\epsilon'_\gamma, \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e) = \frac{4r_e^2(1 + \epsilon'_\gamma)^2}{\cos^3 \vartheta_e} \frac{1}{(a(\epsilon'_\gamma, \vartheta_e) + 2\epsilon'_\gamma)^2} \quad (0.6)$$

$$\times \left[1 - \frac{2}{a(\epsilon'_\gamma, \vartheta_e)} + \frac{2}{a^2(\epsilon'_\gamma, \vartheta_e)} + \frac{2\epsilon_\gamma'^2}{a(\epsilon'_\gamma, \vartheta_e)(a(\epsilon'_\gamma, \vartheta_e) + 2\epsilon'_\gamma)} \right] \quad (0.7)$$

$$\delta_{C,e}(\epsilon'_\gamma, \epsilon_\gamma) := \delta \left(\epsilon_e - \frac{2\epsilon_\gamma'^2}{2\epsilon'_\gamma + a(\epsilon'_\gamma, \vartheta_e)} \right) \quad (0.8)$$

where

$$a(\epsilon'_\gamma, \vartheta_e) := (1 + \epsilon'_\gamma)^2 \tan^2 \vartheta_e + 1. \quad (0.9)$$

Differential cross section for Møller scattering of primary electrons, i.e.,

$$\epsilon_e > (\epsilon'_e - \epsilon_B)/2$$

Literature:(Kawrakew and Rogers,2002) and (Nachtmann,1986)

$$\tilde{\sigma}_M(\epsilon'_e, \epsilon_e, \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e) = \sigma_M(\epsilon'_e, \epsilon_e) \delta_M(\mu_e, \mu_p) \frac{1}{2\pi}, \quad \mu_e = \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e \quad (0.10)$$

with

$$\sigma_M(\epsilon'_e, \epsilon_e) = \frac{2\pi r_e^2 (\epsilon'_e + 1)^2}{\epsilon'_e (\epsilon'_e + 2)} \left[\frac{1}{\epsilon_c^2} + \frac{1}{(\epsilon'_e - \epsilon_e)^2} + \frac{1}{(\epsilon'_e + 1)^2} - \frac{2\epsilon'_e + 1}{(\epsilon'_e + 1)^2 \epsilon_e (\epsilon'_e - \epsilon_e)} \right] \quad (0.11)$$

$$\delta_{M,\delta}(\mu_e, \mu_\delta) = \delta \left(\mu_e - \sqrt{\frac{\epsilon_e \epsilon'_e + 2}{\epsilon'_e \epsilon_e + 2}} \right), \quad \epsilon_e > (\epsilon'_e - \epsilon_B)/2. \quad (0.12)$$

Differential cross section for Møller scattering of secondary electrons, i.e.,

$$\epsilon_e < (\epsilon'_e - \epsilon_B)/2$$

Literature:(Kawrakew and Rogers,2002) and (Nachtmann, 1986)

$$\tilde{\sigma}_{M,\delta}(\epsilon'_e, \epsilon_e, \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e) = \sigma_M(\epsilon'_e, \epsilon_e) \delta_{M,\delta}(\mu_e, \mu_\delta) \frac{1}{2\pi}, \quad \mu_e = \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e \quad (0.13)$$

with

$$\sigma_M(\epsilon'_e, \epsilon_e) = \frac{2\pi r_e^2 (\epsilon'_e + 1)^2}{\epsilon'_e (\epsilon'_e + 2)} \left[\frac{1}{\epsilon_c^2} + \frac{1}{(\epsilon'_e - \epsilon_e)^2} + \frac{1}{(\epsilon'_e + 1)^2} - \frac{2\epsilon'_e + 1}{(\epsilon'_e + 1)^2 \epsilon_e (\epsilon'_e - \epsilon_e)} \right] \quad (0.14)$$

$$\delta_{M,\delta}(\mu_e, \mu_\delta) = \delta \left(\mu_e - \sqrt{\frac{\epsilon_e \epsilon'_e + 2}{\epsilon'_e \epsilon_e + 2}} \right), \quad \epsilon_e < (\epsilon'_e - \epsilon_B)/2. \quad (0.15)$$

Differential cross section for Møller scattering of electrons

Calculation based on (Kawrakew and Rogers, 2002)& (Nachtmann, 1986)

$$\sigma_M^{\text{tot}}(\epsilon_e) = \int_{\epsilon_B}^{(\epsilon_e - \epsilon_B)/2} \sigma_M(\epsilon_e, \epsilon'_e) d\epsilon'_e. \quad (0.16)$$

Two lower limit of integration is due to the fact that the primary electron can only be scattered if at lest the binding energy ϵ_B is transferred to the secondary electron (of a tissue molecule). Besides the evident motivation of this choice based on our model, this is a standard way to avoid singularities in calculating total cross section(Williams M M R, 1979,section 5.1). The upper limit of integration is due to the fact that the primary

electron has larger energy than the secondary electron and that the binding energy ϵ_B was introduced into the scattering processes (usually the upper limit is $\epsilon'_e/2$). One gets

$$\sigma_M^{\text{tot}}(\epsilon_e) = \frac{2\pi r_c^2 (\epsilon_e + 1)^2}{\epsilon_e (\epsilon_e + 2)} \left\{ \frac{1}{\epsilon_B} - \frac{3}{\epsilon_e - \epsilon_B} + \frac{2}{\epsilon_e - \epsilon_B} + \frac{\epsilon_e - 3\epsilon_B}{2(\epsilon_e + 1)^2} + \frac{2\epsilon_e + 1}{\epsilon_e (\epsilon_e + 1)} \right. \\ \left. \times \left[\ln \frac{\epsilon_e + \epsilon_B}{\epsilon_e - \epsilon_B} - \ln \frac{\epsilon_e - \epsilon_B}{\epsilon_B} \right] \right\} \quad (0.17)$$

Differential cross section for Mott scattering of electron

Literature: (Mott and Massey, 1965) & (Lehmann, 1977)

$\alpha \approx 1/137$ is the fine structure constant, Z is the atomic number of the irradiate medium. Z depends on r to account for heterogeneous media.

$$\sigma_{\text{Mott}}(r, \epsilon_e, \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e) = \frac{Z^2(r) r_c^2 (mc^2)^2}{4p^2 c^2 \beta^2 \sin^4(\frac{\vartheta_e}{2})} \left[1 - \beta^2 \sin^2 \frac{\vartheta_e}{2} + Z\pi\alpha\beta \sin \frac{\vartheta_e}{2} (1 - \sin \frac{\vartheta_e}{2}) \right] \quad (0.18)$$

$$\approx \frac{Z^2(r) r_c^2 (mc^2)^2}{4p^2 c^2 \beta^2 \sin^4(\frac{\vartheta_e}{2})} \left[1 - \beta^2 \sin^2 \frac{\vartheta_e}{2} \right], \quad (0.19)$$

with $\beta^2 = \frac{\epsilon_e(\epsilon_e+2)}{(\epsilon_e+1)^2}$. The most approximation is justified, because in the energy range studied here and for typical low- Z media like water only small errors are made.

To avoid the singularity at $\vartheta_e = 0$ a screening paramcter η can be introduced in [?] that models the screening effect of the electron and the atomic shell:

$$\sigma_{\text{Mott}}(r, \epsilon_e, \mathbf{\Omega}'_e \cdot \mathbf{\Omega}_e) = \frac{Z^2(r) r_c^2 (1 + \epsilon_e)^2}{4[\epsilon_e(\epsilon_e + 2)]^2 (1 + 2\eta(r, \epsilon_e) - \cos \vartheta_e)^2} \left[1 - \frac{\epsilon_e(\epsilon_e + 2)}{(1 + \epsilon_e)^2} \sin^2 \frac{\vartheta_e}{2} \right] \quad (0.20)$$

with

$$\eta(r, \epsilon_e) = \frac{\pi^2 \alpha^2 Z^{\frac{2}{3}}(r)}{\epsilon_e (\epsilon_e + 2)}. \quad (0.21)$$

Total cross section for Mott scattering of electron

$$\sigma_{\text{Mott}}^{\text{tot}}(r, \epsilon_e) = \frac{\pi (Z(r) r_c)^2}{\epsilon_e (\epsilon_e + 2)} \left[\frac{(\epsilon_e + 1)^2}{(\pi\alpha)^2 Z^{\frac{2}{3}}(r) (1 + \eta(r, \epsilon_e))} + \frac{1}{1 + \eta(r, \epsilon_e)} \right. \\ \left. + \ln \eta(r, \epsilon_e) - \ln(1 + \eta(r, \epsilon_e)) \right]. \quad (0.22)$$

Appendix B

The Møller stopping power S_M

Insted of using the definition of Pomraning(1992)it is convenient to use the standard definition of the stopping power:

$$S_M(r, \epsilon_c) = \rho_c(r) \int_{\epsilon_B}^{(\epsilon_c - \epsilon_B)/2} \epsilon'_e \sigma_M(\epsilon_c, \epsilon'_e) d\epsilon'_e, \quad (0.1)$$

Of course both definitions are equivalent.

This integral can be evaluated analytically, too, and one gets

$$S_M(r, \epsilon_c) = \frac{2\pi r_e^2 \rho_e(r) (\epsilon_c + 1)^2}{\epsilon_c (\epsilon_c + 2)} \left[\frac{2\epsilon_c}{\epsilon_c - \epsilon_B} + 2 \ln \frac{\epsilon_c - \epsilon_B}{2} - \ln \epsilon_B - \frac{\epsilon}{\epsilon - \epsilon_B} - \ln(\epsilon - \epsilon_B) \right. \\ \left. + \frac{1}{2(\epsilon_c + 1)^2} \left(\frac{(\epsilon_c - \epsilon_B)^2}{4} - \epsilon_B^2 \right) + \frac{2\epsilon_c + 1}{(\epsilon_c + 1)^2} \left(\ln \frac{\epsilon_c - \epsilon_B}{2} - \ln(\epsilon_c - \epsilon_B) \right) \right]. \quad (0.2)$$