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Some Results of Stationary Axisymmetric E-M Equations

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Abstract : *The purpose of this paper is to derive a number of results in connection with the axisymmetric sourceless stationary Einstein-Maxwell equations. Among the known rotating solutions of the vacuum Einstein equations are the Lewis (1932)[1] and Papapetrou (1947)[2] solutions, each class depending on a single harmonic function. Bonor (1973)[3] found a class of exact solutions of the E-M equations that are analogous to the Papapetrou solutions. In this communication, we find a class of exact rotating solutions of the vacuum E-M equations depending on a harmonic function and corresponding to the Lewis solutions. Like the latter these new solutions have singularities extending to infinity along the axis of symmetry and hence cannot represent realistic sources, but can represent the exterior field of infinite axially symmetric rotating electromagnetic sources. The new solutions are derived from the Lewis solutions by the application of a method due to Ernst(1968)[4].*

1. Introduction and the Field Equations :

The field equations of the Einstein-Maxwell theory, in units in which the gravitational constant and the velocity of light are equal to unity, are as follows:

$$R_{\mu\upsilon} = -8\pi E_{\mu\upsilon} = 2F^{\alpha}_{\mu}F_{\upsilon\alpha} - \frac{1}{2}g_{\eta\upsilon}F_{\alpha\beta}F^{\alpha\beta} \tag{1.1}$$

$$F_{\mu\upsilon;\sigma} + F_{\upsilon\sigma;\mu} + F_{\sigma\mu;\upsilon} = 0, \quad F^{\mu\upsilon}_{;\upsilon} = J^{\mu} = 0 \tag{1.2}$$

where $E_{\mu\upsilon}$ is the electromagnetic energy-momentum tensor and J^{μ} the four current which is zero in the exterior field. The sign convention for $R_{\mu\upsilon}$ is the same as in Bonnor (1973) and Islam (1976 a,b). The electromagnetic field tensor $F_{\mu\upsilon}$ is given in terms of a four vector potential A_{μ} as follows:

$$F_{\mu\upsilon} = \frac{\partial A_{\mu}}{\partial x^{\upsilon}} - \frac{\partial A_{\upsilon}}{\partial x^{\mu}} \tag{1.3}$$

in which case the first of (1.2) is satisfied identically.

The axisymmetric stationary metric for the present case is put in the following form

$$ds^2 = f(dt - wd\theta)^2 - f^{-1}\rho^2 d\theta^2 - e^{\beta}(dp^2 + dz^2) \tag{1.4}$$

where f, w and β are functions of ρ and z.

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Following Bonnor (1973) and Islam (1985), and writing $\left(x^0, x^1, x^2, x^3\right)=\left(t, \rho, z, \theta\right)$ the vector potential $\left(A_0, A_1, A_2, A_3\right)$ in terms of scalar fields Φ *and* Ψ and the metric functions f and w are as follows:

$$A_0 = \Psi, A_1 = A_2 = 0, \frac{\partial A_3}{\partial \rho} = w\Phi_\rho + \rho f^{-1}\Psi_z$$

$$\frac{\partial A_3}{\partial z} = w\Phi_z - \rho f^{-1}\Psi_\rho \tag{1.5}$$

where $\Phi_\rho = \frac{\partial \Phi}{\partial \rho}$ etc. and the field equations are

$$\nabla^2 f - f^{-1}(f_\rho^2 + f_z^2) = 2\left(\Phi_\rho^2 + \Phi_z^2 + \Psi_\rho^2 + \Psi_z^2\right) - \rho^{-2} f^3\left(w_\rho^2 + w_z^2\right) \tag{1.6a}$$

$$\nabla^2 w - 2\rho^{-1} w_\rho = -2f^{-1}(f_\rho w_\rho + f_z w_z) + 4\rho f^{-2}\left(\Psi_z \Phi_\rho - \Psi_\rho \Phi_z\right) \tag{1.6b}$$

$$\nabla^2 \Phi = f^{-1}(f_\rho \Phi_\rho + f_z \Phi_z) - \rho^{-1} f\left(w_\rho \Psi_z - w_z \Psi_\rho\right) \tag{1.6c}$$

$$\nabla^2 \Psi = f^{-1}(f_\rho \Psi_\rho + f\Psi_z) - \rho^{-1} f\left(w_\rho \Phi_z - w_z \Phi_\rho\right) \tag{1.6d}$$

where $\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} + \rho^{-1}\frac{\partial}{\partial z}$ and for β the field equations are

$$\beta_\rho = -f^{-1} f_\rho + \frac{1}{2}\rho f^{-2}\left(f_\rho^2 - f_z^2\right) + 2\rho f^{-1}\left(\Phi_z^2 - \Phi_\rho^2 + \Psi_z^2 - \Psi_\rho^2\right) \tag{1.7a}$$

$$+\frac{1}{2}\rho^{-1} f^2\left(w_z^2 - w_\rho^2\right)$$

$$\beta_z = -f^{-1} f_z + \rho f^{-2} f_\rho f_z - 4\rho f^{-1}\left(\Phi_\rho \Phi_z + \Psi_\rho \Psi_z\right) - \rho^{-1} f^2 w_\rho w_z \tag{1.7b}$$

the consistency of which is assured by (1.6a-1.6b).

To reduce these equations in Ernst form, a potential u was considered, the integrability of whose existence is given by Islam (1985)

$$\rho^{-1} f^2 w_\rho - 2(\phi\psi_z - \psi\phi_z) = u_z, \rho^{-1} f^2 w_z + 2\left(\phi\psi_\rho - \psi\phi_\rho\right) = -u_\rho \tag{1.8}$$

Eliminating w by u the equations (1.6a-1.6d) are transformed to the following equations:

$$f\nabla^2 f - f_\rho^2 - f_z^2 = 2f\left(\phi_\rho^2 + \phi_z^2 + \psi_\rho^2 + \psi_z^2\right) - \left(2\phi\psi_\rho - 2\psi\phi_\rho + u_\rho\right)^2 \tag{1.9a}$$

$$-\left(2\phi\psi_z - 2\psi\phi_z + u_z\right)^2$$

$$f\nabla^2 u = 4\phi\psi\left(\phi_\rho^2 + \phi_z^2 - \psi_\rho^2 - \psi_z^2\right) + 4\left(\psi^2 - \phi^2\right)\left(\phi_\rho \psi_\rho + \phi_z \psi_z\right)$$

$$-2\psi\left(\psi_\rho u_\rho + f_z \psi_z\right) - 2\phi\left(\phi_\rho u_\rho + \phi_z u_z\right) + 2\left(f_\rho u_\rho + f_z u_z\right) \tag{1.9b}$$

$$-2\psi\left(f_\rho \varphi_\rho + f_z \varphi_z\right) + 2\varphi\left(f_\rho \psi_\rho + f_z \psi_z\right)$$

$$f\nabla^2\varphi = f_{,\rho}\varphi_{,\rho} + f_{,z}\varphi_{,z} - 2\phi\left(\psi_{\rho}^2 + \psi_{z}^2\right) + 2\psi\left(\phi_{,\rho}\psi_{,\rho} + \phi_{,z}\psi_{,z}\right) - \left(u_{,\rho}\psi_{,\rho} + u_{,z}\psi_{,z}\right) \qquad (1.9c)$$

$$f\nabla^2\psi = f_{,\rho}\psi_{,\rho} + f_{,z}\psi_{,z} + 2\phi\left(\phi_{,\rho}\psi_{,\rho} + \phi_{,z}\psi_{,z}\right) - 2\psi\left(\varphi_{\rho}^2 + \varphi_{z}^2\right) + \left(u_{,\rho}\varphi_{,\rho} + u_{,z}\varphi_{,z}\right) \qquad (1.9d)$$

where (1.9b) has been obtained by eliminating w from (1.6b) and using (1.9c,d)

Then defining the complex functions E and F as follows

$$E = f - \Phi^2 - \Psi^2 + iu, F = \Phi + i\Psi \qquad (1.10)$$

equations (1.9a-d) are combined in the following complex equations.

$$\left(\operatorname{Re} E + |F|^2\right)\nabla^2 E = E_{\rho}^2 + E_{z}^2 + 2F^*\left(E_{,\rho}F_{,\rho} + E_{,z}F_{,z}\right)$$

$$\left(\operatorname{Re} E + |F|^2\right)\nabla^2 F = E_{,\rho}F_{,\rho} + E_{,z}F_{,z} + 2F^*\left(F_{\rho}^2 + F_{z}^2\right) \qquad (1.11b)$$

which are the Ernst's form(1968) of E-M equations, where a star "*" denotes complex conjugation.

Following Islam (1985)[6] the complex functions E and F can be transformed by the Gauge transformations

$$E' = E - |C|^2 - 2C^*F + iC' \qquad (1.12)$$

$$F' = F + C$$

where C and C' are respectively a complex and a real constant.

If E is an analytic function of F, then this function can be considered linear and under the boundary conditions $E \to 1$ and $F \to 0$ at infinity, it can be taken as

$$E = 1 - 2Q^{-1}F \qquad (1.13)$$

where $Q = q + iq'$, is a complex constant .

Three cases arise according as $|Q| < 1, = 1$, or > 1

Case (I): $|Q| < 1, \xi$ is defined by

$$E = \frac{\alpha\xi - 1}{\alpha\xi + 1}, \alpha = (1 - QQ^*)^{\frac{1}{2}} \qquad (1.14)$$

The equations (1.11a, b) reduce to the single equation

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*\left(\xi_{\rho}^2 + \xi_{z}^2\right) \qquad (1.15)$$

This is the pure Einstein equation, that is, equations (1.9 a,b) with $\Phi = \Psi = 0$ can be reduced to the form (1.15) where ξ is given in terms of f and the potential u by the following (Ernst 1968)

$$\xi = \frac{(1+f+iu)}{(1-f-iu)} \tag{1.16}$$

Thus any solution of the stationary axisymmetric Einstein equations gives a corresponding solution of the E M equations through the above procedure. We do not consider caseII & case III.

2. Derivation of the new solution

The Lewis solutions are given as follows (1932)

$$f = \rho\left(a_1^2 e^{\eta} - b_1^2 e^{-\eta}\right) \equiv f', \quad w = \frac{\left(-a_1 a_2 e^{\eta} + b_1 b_2 e^{-\eta}\right)}{a_1^2 e^{\eta} - b_1^2 e^{-\eta}} \tag{2.01}$$

where $a_1 b_2 - a_2 b_1 = 1$

and a_1, a_2, b_1 and b_2 are constants and η is a harmonic function satisfying

$$\eta_{\rho\rho} + \eta_{zz} + \rho^{-1}\eta_{\rho} = 0 \tag{2.02}$$

The potential u is given as follows:

$$u_z = 2a_1 b_1 \rho \eta_{\rho} \equiv u'_z, \quad u_{\rho} = -2a_1 b_1 \rho \eta_z \equiv u'_{\rho} \tag{2.03}$$

An alternative form of u' is

$$u' = \rho \eta'_{\rho}, \quad \eta'_z = 2a_1 b_1 \eta \tag{2.04}$$

so that η' is also harmonic.

Thus ξ given by $\xi = \dfrac{1+f'+iu'}{1-f'-iu'} \equiv \xi'(say)$ (2.05)

satisfies (1.15) and E given by (1.14) and the following equation

$$E = \frac{\alpha\xi' - 1}{\alpha\xi' + 1} \equiv X + iY$$

constitute a solution of the E-M equations.

The functions f, w, Φ and Ψ can be obtained explicitly as follows :

$$X = \frac{\left\{(\alpha^2-1)(1+f'^2) + 2(\alpha^2+1)f' + (\alpha^2-1)u'^2\right\}}{\left[\{(\alpha+1)+(\alpha-1)f'\}^2 + (\alpha-1)^2 u'^2\right]} \tag{2.06a}$$

$$Y = \frac{4\alpha u'}{\left[\{(\alpha+1)+(\alpha-1)f'\}^2 + (\alpha-1)^2 u'^2\right]} \tag{2.06b}$$

where u' is given by (2.03) and f by (2.01) and (2.03)

i.e. $$f' = \rho\left(a_1^2 e^{\frac{\eta'_z}{2a_1 b_1}} - b_1^2 e^{-\frac{\eta'_z}{2a_1 b_1}}\right) \tag{2.06c}$$

With the use of (1.14) and (2.06), we solve for Φ and Ψ as follows:

$$\Phi = \frac{1}{2}\{q(1-X) + q'Y\}, \Psi = \frac{1}{2}\{q'(1-X) - qY\} \tag{2.07}$$

Relation (1.14) also implies

$$f=\frac{1}{4}\left(q^2+q'^2\right)\left\{(1-X)^2+Y^2\right\}+X \tag{2.08}$$

The function w is given by one of the following equations:

$$w_\rho=\rho f^{-2}\left[-\frac{1}{2}\left(q^2+q'^2\right)\left(YX_z-XY_z+Y_z\right)+Y_z\right] \tag{2.09a}$$

$$w_z=\rho f^{-2}\left[\frac{1}{2}\left(q^2+q'^2\right)\left(YX_\rho-XY_\rho+Y_\rho\right)-Y_\rho\right] \tag{2.09b}$$

where f is given by (2.08).

After substitution for X and Y from (2.06a, b) we can reduce the functions f and w to the following form ,in which the dependence of these functions on ρ and the harmonic function η' is apparent.

$$f=\frac{4\alpha^2\rho H}{\left[\left\{(\alpha+1)+(\alpha-1)\rho H\right\}^2+(\alpha-1)^2\rho^2\eta_\rho'^2\right]} \tag{2.10a}$$

$$4\alpha^2 w_\rho=\left\{(\alpha+1)^2H^{-2}-(\alpha-1)^2\rho^2+(\alpha-1)^2\rho^2H^{-2}\eta_\rho'^2\right\}\eta'_{\rho z}$$
$$+2(\alpha-1)^2\rho^2H^{-1}G\eta'_\rho\eta'_{zz} \tag{2.10b}$$

$$4\alpha^2 w_z=\left\{(\alpha+1)^2H^{-2}-(\alpha-1)^2\rho^2+(\alpha-1)^2\rho^2H^{-2}\eta_\rho'^2\right\}\eta'_{zz}$$
$$-2(\alpha-1)^2\rho\eta'_\rho\left(1+\rho H^{-1}G\eta'_{\rho z}\right) \tag{2.10c}$$

where $H=a_1^2e^{\eta'_z/2a_1b_1}-b_1^2e^{-\eta'_z/2a_1b_1}$

$$G=\frac{1}{2}\left(\frac{a_1}{b_1}\right)e^{\eta'_z/2a_1b_1}+\frac{1}{2}\left(\frac{b_1}{a_1}\right)e^{-\eta'_z/2a_1b_1} \tag{2.10d}$$

i.e. $f'=\rho H$ and G is the derivative of H with respect to η'_z. In Appendix A we verify the consistency of (2.10 b,c). The solution thus depend on a harmonic function and five arbitrary constants, three of a_1, b_1, a_2, b_2 and q, q'.

Physical Interpretation

We give a few arguments to show that the new metric has similar singularity properties to the Lewis metric, namely, that it has singularities along the axis of symmetry extending to infinity, and hence cannot represent the exterior field of a bounded source. A realistic bounded rotating source produces a field that has the following form at infinity

(see, e.g. Islam 1985, p.31)[7]:

$$F=1+\frac{m'}{r}+\ldots,\quad w=\frac{a'\rho^2}{r^3}+\ldots,\quad r^2=\rho^2+z^2 \tag{2.11}$$

where the dots represent terms that vanish at infinity faster than those preceding the dots, and m', a' are constants related to the mass and angular momentum respectively of the source. For the Lewis solution (2.01) the condition that w tends to zero at infinity means, quite independent of the manner it tends to zero, that η tends to a constant η_0 given by

$$-a_1a_2e^{\eta_0}+b_1b_2e^{\eta_0}=0,\ a_1^2e^{\eta_0}-b_1^2e^{-\eta_0}\neq 0 \tag{2.12}$$

This implies that f behaves like a constant times ρ at infinity which is inadmissible for realistic sources. Also, in the neighborhood of the axis of symmetry, the coefficients of $d\theta^2$, given by $fw^2-f^{-1}\rho^2$, does not behave like ρ^2 as $\eta\to\eta_0$, but like ρ, implying that there is a singularity on the axis of symmetry. One could replace η by $n\log\rho+\eta$ where n is a constant, since $\log\rho$ is also harmonic, but the situation remains essentially the same.

Similar considerations can be applied to the new metric given by (2.07) and w tending to zero at infinity but does not necessarily imply the latter. Then the function E (see 2.10d) also remains finite. Eq^n (2.10 a) then implies that at infinity in the neighborhood of $\rho=0$, the function f behaves like ρH , which violates the behavior (2.10d). It does not seem possible to find a harmonic function η' which will avoid these unphysical properties of the metric along $\rho=0$, which suggests that there is a singularity along this axis. Again the situation is not improved by adding a constant multiple of $\log\rho$ to η' or η'_{z_0} .

A rigorous discussion of the behavior of the metric involves the computation of the curvature invariants, one of which is $F_{\mu\nu}F^{\mu\nu}$. However, since in this invariant the functions ρ and w occur explicitly, its behavior is not immediately clear as these functions are given in terms of integral , e.g., w is proportional to the integral with respect to ρ and z of the right hand sides of (2.10 b,c) respectively.

The new solutions could represent the exterior electromagnetic and gravitational field of charged sources extending to infinity along the axis of symmetry. As mentioned earlier, a class of exact interior solutions representing such sources has been found by Islam (1977)[8].

Appendix A

We verify the consistency of (2.10 b,c) i.e., we show that the derivative with respect to z of the right hand side of (2.10 b) equals the derivative with respect to ρ of the right hand side of (2.10c). This condition is as follows:

$$\begin{bmatrix}\left\{-2(\alpha+1)^2H^{-3}H_z+(\alpha-1)^2\rho^2\left(2H^{-2}\eta'_\rho\eta'_{\rho z}-2H^{-3}H_z\eta'^2_\rho\right)\right\}\eta'_{\rho z}\\ +2(\alpha-1)^2\rho^2\left(-H^{-2}GH_z\eta'_\rho\eta'_{zz}+H^{-1}G_z\eta'_\rho\eta'_{zz}\right)\end{bmatrix} \tag{A.1}$$

$$+\left[\begin{array}{l}\left\{\begin{array}{l}2(\alpha+1)^2 H^{-3}H_{\rho}-(\alpha-1)^2\rho H^{-2}\eta_{\rho}'^{2}- \\ (\alpha-1)^2\rho^2\left(2H^{-2}\eta_{\rho}'\eta_{\rho\rho}'-2H^{-3}H_{\rho}\eta_{\rho}'^{2}\right)\end{array}\right\}\eta_{zz}' \\ +2(\alpha-1)^2\rho^2\eta_{\rho}'\left(H^{-1}G_{\rho}\eta_{\rho z}'-H^{-2}GH_{\rho}\eta_{\rho z}'\right)\end{array}\right]=0,$$

where H and G are defined by ((2.10d), and where a number of terms have already been cancelled, including all terms involving third derivatives of, by virtue of being harmonic. For the remaining terms use the following identities which follow from the definition of H and G.

$$H_{\rho}=G\eta_{\rho z}',\ H_{z}=G\eta_{zz}',\ 4a_1^2b_1^2G_{\rho}=H\eta_{\rho z}',$$
$$4a_1^2b_1^2G_{z}=H\eta_{zz}',\ 4a_1^2b_1^2\left(1-G^2\right)=H^2 \tag{A.2}$$

Consider first terms with a factor. With the use of (A.2) these terms can be shown to vanish. Next consider terms with factor . This factor cancels with a factor that emerges by virtue of the last relation in (A.2). The remaining terms can easily be shown to cancel each other.

Consider a system of coupled partial differential equations involving derivatives no higher than the second, in which the second derivatives occur only in the Laplacian operator. If one is seeking solutions of these equations, either approximate or exact, it is often fruitful to try solutions in terms of harmonic functions. In this connection, for the axismmetric case, it is useful to have two expression I and J involving the derivatives of one or more harmonic functions and the variable ρ (in usual cylinder polar coordinates) such that

$$I_{\rho}=J_{z} \tag{A.3}$$

A possible guess for one of the unknown is the following

$$w_{\rho}'=J,\ w_{z}'=I \tag{A.4}$$

The consistency of which is guaranteed by (A.3). Such examples occur in (Islam, 1985, p-72). In the following I and J are used in Islam (1976a).

$$I=-\rho\left(U_{\rho}V_{z}+U_{z}V_{\rho}\right),\ J=\rho\left(U_{z}V_{z}-U_{\rho}V_{\rho}\right) \tag{A.5}$$

where U and V are harmonic functions. The examples (A.5) with $U=V$ occurs in the static axisymmetric Weyl(1917) solutions of the pure Einstein equations and also in the electrostatic Weyl solutions (see also Islam 1976[5]a, b 1978). In this examples the variable occurs in I and J as overall factor. It is non-trivial to determine functions I and J for which this is not the case. In eqs. (2.10d) we have such an example, which has arisen naturally when Ernst's method has been applied to the Lewis solution. This example may lead to other similar ones, which may prove useful in other contexts where one is looking for solutions in terms of harmonic functions.

References

1. Lewis,T.,(1932) Proc. Roy. Soc. Lond. A 136,176.
2 PAPAPETROU, A., (1947). Proc. Roy. Irish Acad. 51, 191.
3. BONNOR, W. B. (1973). Commun. Math. Phys. 34, 77.
4. ERNST, F. J. (1968). Phys. Rev. 168, 1415.
5. ISLAM, J. N. (1976a). Math. Proc. Cambridge Phil. Soc. 79, 161.
6. ISLAM, J. N. (1976b). Gen. Rel. Grav. 7, 669.
7. ISLAM, J. N. (1985). Rotating Fields in General Relativity, Cambridge University Press.
8. ISLAM, J. N. (1977). Proc. Roy, Soc. 353A, 523.
9. ISLAM, J. N. (1978). Gen. Rel. Grav. 9, 687.
10. WEYL, H. (1917). Ann. Phys. Leipzig, 54, 117.